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H_∞ Inverse Optimal Attitude-Tracking Control of Rigid Spacecraft

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The attitude trajectory tracking control problem for a rigid spacecraft with external disturbances is addressed using the robust inverse optimal-control method. The proposed feedback-control law is optimal with respect to a meaningful cost functional involving tracking errors, control efforts, and extended disturbances, and the associated Lyapunov function satisfies a Hamilton–Jacobi–Isaacs partial differential equation. The controller is H_∞ optimal with respect to extended disturbances. The performance limitation of the inverse optimal feedback controller is analyzed and guidelines for the selection of the controller gains are established. Numerical simulations are performed to demonstrate the effectiveness of the proposed control algorithm and the tuning guidelines.

Nomenclature

$\|A\|$	=	induced 2-norm of the matrix $A \in \mathcal{R}^{n\times n}$, $\|A\| = \sqrt{[\lambda_{\max}(A^T A)]}$
A^T	=	transpose of A
$\|a\|$	=	Euclidean norm of the vector $a \in \mathcal{R}^n$, $\|a\| = \sqrt{(a^T a)}$
$d, \hat{d}$	=	external disturbance and extended disturbance, respectively
J	=	inertia matrix of the spacecraft, $J = J^T > 0$
$L_f V$	=	Lie derivative of the Lyapunov function $V(x)$ with respect to $f(x)$, $L_f V = \dfrac{\partial V(x)}{\partial x} f(x)$
$\mathcal{L}_2[0, \infty)$	=	linear space consisting of square integrable $\mathcal{R}^m$-valued functions; that is, $v \in \mathcal{L}_2[0, \infty)$ implies that $\int_0^\infty v(t)^T v(t)\,\mathrm{d}t$ is finite
q, q_c, q_e	=	actual quaternion of spacecraft, target quaternion, and quaternion error, respectively; $q_e = [\epsilon^T, \eta]^T$
$\mathcal{R}$	=	real space
$\mathcal{S}^3, \mathcal{TS}^3$	=	unit sphere in $\mathcal{R}^4$ and its tangent bundle, respectively
w, w_c, w_e	=	actual angular velocity of spacecraft, desired angular velocity, and rate error, respectively; $w_e = w - w_c$
$x, \tilde{x}$	=	system state $x = [\epsilon^T, \eta, w_e^T]^T$ and tracking errors $\tilde{x} = [\epsilon^T, w_e^T]^T$
λ_i	=	$\lambda_i = \lambda_{\max}(J^{-1})$, the maximum eigenvalue of J^{-1}
λ_j	=	$\lambda_j = \lambda_{\max}(J)$, the maximum eigenvalue of inertia matrix J

I. Introduction

THE present generation of spacecraft requires attitude-control systems to provide attitude-maneuver, tracking, and pointing capabilities, whereas the equations that govern large-angle maneuvers in the presence of exogenous disturbances are nonlinear and highly coupled. Thus, control system design must consider nonlinear dynamics.

Various nonlinear control algorithms have been proposed for solving the attitude-control problem. These include nonlinear feedback control,[1] feedback linearization,[2] variable-structure sliding control,[3] linearly bounded control,[4] nonlinear adaptive control,[5] and nonlinear H_∞ control.[6] Because its inherent robustness with respect to disturbances and model uncertainty, the nonlinear H_∞ optimal-control method[7] is a potential approach to solving the nonlinear attitude-control problem. However, the practical application of nonlinear H_∞ optimal control is still an open problem due to the difficulty of solving the associated Hamilton–Jacobi–Isaacs (HJI) partial differential equation. Many methods have been proposed in an attempt to solve the HJI equation. One of the methods that provide approximate solutions is the state-dependent Riccati equation method,[8] though only suboptimality and local stability can be guaranteed. Algebraic and geometric tools[9–11] were employed to study a particular H_∞ suboptimal control problem by solving the associated HJI partial differential inequality. Note that the derived H_∞ suboptimal control laws[9–11] were designed to solve the attitude-stabilizing control problem and the $\mathcal{L}_2$-gain γ was restricted to be larger than a certain value. A power series solution[12] of the HJI inequality was proposed for the H_∞ suboptimal control of wing rock motions by representing the state vector as a series of closed-loop Lyapunov functions. The concept of extended disturbances, including system error dynamics, was introduced into robotics by Park and Chung[13] to solve the HJI equation. Besides these attempts to find analytical solutions, a numerical approach[14] was proposed as a systematic way to find the numerical solution of the HJI equation.

In this paper we follow an alternative approach to derive H_∞ optimal feedback control laws for the nonlinear attitude-tracking control problem of rigid spacecraft with external disturbances. We employ the robust inverse optimal-control approach,[15,16] which circumvents the task of solving the HJI partial differential equation and results in a controller that is optimal with respect to a set of cost functionals involving tracking errors, control efforts, and disturbances and achieves disturbance attenuation. The robust inverse optimality approach requires knowledge of a control Lyapunov function and a stabilizing control law for an auxiliary nonlinear system. The first application of this approach to the attitude-control problem was presented in Ref. 17, in which a nonlinear inverse optimal feedback controller based on Rodrigues parameters was proposed for the attitude-stabilizing control problem of a rigid spacecraft in the absence of exogenous disturbances. The optimal feedback-control

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law[17] is a regional control algorithm because the attitude representation using the Rodrigues parameters has a singularity. Also, only the stabilizing-control problem and not the attitude-tracking problem was considered in Ref. 17.

Based on attitude representation using unit quaternions, which is globally nonsingular, the attitude-tracking control problem subject to disturbances is studied in this paper. The main contributions of this paper relative to other related works are as follows:

1) With the introduction of the concept of extended disturbance, the inverse optimality approach is applied to solve the attitude-tracking control problem subject to exogenous disturbances. This approach has been used in robot control[13,18] for the trajectory-tracking problem of Euler–Lagrangian systems but not in the attitude-control literature for the attitude-tracking problem of spacecraft with external disturbances.

2) The proposed inverse optimal controller is also H_∞ optimal with respect to the extended disturbance, thus achieving H_∞ disturbance attenuation. By comparison, the H_∞ suboptimal controllers in Refs. 6 and 9–11 were designed for the attitude-stabilization problem but not the attitude-tracking problem. Furthermore, the $\mathcal{L}_2$ gains γ of those H_∞ suboptimal controllers were restricted to be larger than certain values. Under the H_∞ inverse optimal controller in this paper, the $\mathcal{L}_2$ gain γ of the closed-loop system is required to be positive only. It can be chosen to be sufficiently small to achieve any level of H_∞ disturbance attenuation at a cost of a larger control effort.

3) The proposed controller is a proportional-derivative (PD) controller, and tuning rules are established in this paper for selecting controller gains based on performance analysis. PD controllers were designed in Refs. 1, 6, and 9–11 and conditions were given for the PD control gains. However, how to tune the control gains on the basis of performance analysis was not analytically stated there.

The rest of the paper is organized as follows. In the next section, we will briefly review some preliminaries of the nonlinear H_∞ control method and the inverse optimal-control approach. In Sec. III, we formulate the nonlinear attitude-tracking control problem of spacecraft. In Sec. IV, a state-feedback H_∞ inverse optimal feedback controller is derived using the robust inverse optimal-control method. Performance estimates for the inverse optimal controller are analyzed and selection guidelines for the controller gains are established in Sec. V. Numerical simulations in Sec. VI illustrate the theoretical results of the paper. Finally, conclusions follow in Sec. VII.

II. Preliminaries

In this section, we will survey some standard results of nonlinear H_∞ control theory[7] and the inverse optimal control method.[16]

A function $\alpha : \mathcal{R}^+ \to \mathcal{R}^+$ is of class $\mathcal{K}$ if it is continuous, positive definite, and strictly increasing. It is of class $\mathcal{K}_\infty$ if it is also unbounded. A function $\beta : \mathcal{R}^+ \times \mathcal{R}^+ \to \mathcal{R}^+$ is of class $\mathcal{KL}$ if, for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed $s \geq 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$.

Consider a nonlinear system of the form

$$\dot{x} = f(x) + g_1(x)d + g_2(x)u, \qquad y = h(x) \tag{1}$$

where x is the state vector, d is the exogenous disturbance to be rejected, u is the control input, and y is the penalized output signal. We assume that $f(x)$, $g_1(x)$, $g_2(x)$, and $h(x)$ are smooth functions and $x = 0$ is the equilibrium point of the nonlinear system; that is, $f(0) = h(0) = 0$. Also, the disturbance d is assumed to be bounded with a known bound. Hence,

$$\int_0^T |d(t)|^2 \,\mathrm{d}t < \infty$$

for all finite $T \geq 0$.

The nonlinear state-feedback H_∞ control problem is to find a state-feedback control $u = k(x)$ for Eqs. (1), with $k(0) = 0$, such that the $\mathcal{L}_2$ gain from the disturbance d to the block vector of outputs y and inputs u is not larger than γ, that is, such that there exists a positive function $K(x) \geq 0$ with $K(0) = 0$ such that

$$\int_0^T \left(|y(t)|^2 + u^T R_2(x)u\right) \mathrm{d}t \leq \gamma^2 \int_0^T |d(t)|^2 \,\mathrm{d}t + K(x_0) \tag{2}$$

is satisfied for all $T \geq 0$ and for any initial condition $x(0) = x_0$ of (1), where $R_2(x)$ is symmetric and positive definite for all x. The H_∞ optimal problem is to find, if it exists, the smallest value γ^* of such an $\mathcal{L}_2$ gain γ.

Lemma 1: Consider the nonlinear system (1). Let the constant $\gamma > 0$ and the matrix $R_2(x) = R_2^T(x) > 0$ for all x. Suppose that there exists a smooth solution $V(x) \geq 0$ with $V(0) = 0$ to the Hamilton–Jacobi–Issacs partial differential equation

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x}\left[\frac{1}{\gamma^2} g_1(x) g_1^T(x) - g_2(x) R_2^{-1}(x) g_2^T(x)\right]\left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{4} h^T(x) h(x) = 0 \tag{3}$$

or to the Hamilton–Jacobi–Issacs partial differential inequality

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x}\left[\frac{1}{\gamma^2} g_1(x) g_1^T(x) - g_2(x) R_2^{-1}(x) g_2^T(x)\right]\left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{4} h^T(x) h(x) \leq 0 \tag{4}$$

Then the closed-loop system for the feedback

$$u = -2R_2^{-1}(x) g_2^T(x) \left(\frac{\partial V}{\partial x}\right)^T \tag{5}$$

has $\mathcal{L}_2$ gain less than or equal to γ from the disturbance d to the block vector of outputs h and control inputs u. The "worst-case" disturbance is given by

$$d^* = \frac{2}{\gamma^2} g_1^T(x) \left(\frac{\partial V}{\partial x}\right)^T$$

Proof: The lemma is a direct result of a theorem of Van der Schaft.[7] From Eq. (3),

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} g_1(x) d + \frac{\partial V}{\partial x} g_2(x) u \\ &= \left| R_2^{-\frac{1}{2}} \left(L_{g_2} V\right)^T - \frac{1}{2} R_2^{\frac{1}{2}} u \right|^2 - \frac{1}{\gamma^2} \left| \left(L_{g_1} V\right)^T - \frac{\gamma^2}{2} d \right|^2 \\ &\quad - \frac{1}{4} |h|^2 - \frac{1}{4} u^T R_2 u + \frac{\gamma^2}{4} |d|^2\end{aligned}$$

Choosing u as in Eq. (5) and integrating with respect to t from 0 to ∞, we conclude that relation (2) is satisfied with $K(x_0) = 4V(x_0)$ and thus the $\mathcal{L}_2$ gain is not larger than γ. □

In general, the HJI partial differential inequality (4) only guarantees a suboptimal solution. The main and most challenging task in solving the nonlinear H_∞ control problem is to find a smooth positive function $V(x)$ satisfying the HJI equation (3) or the HJI inequality (4). However, it is in general very difficult to solve the HJI equation (3) or the HJI inequality (4).

Compared with nonlinear H_∞ control, the inverse optimal method[16] solves the nonlinear optimal-assignment problem with respect to a meaningful cost functional without solving the HJI partial differential equation explicitly. For completeness of this paper, we quote the following theorem of the inverse optimal method.

Theorem 1 (Ref. 16): Consider the nonlinear control system (1) and its auxiliary system

$$\dot{x} = f(x) + g_1(x)\ell_\rho\left(2\left|L_{g_1} V\right|\right)\left[\left(L_{g_1} V\right)^T \big/ \left|L_{g_1} V\right|^2\right] + g_2(x)u \tag{6}$$

where $V(x)$ is a Lyapunov function candidate and ρ is a class $\mathcal{K}_\infty$ function whose derivative ρ' is also a class $\mathcal{K}_\infty$ function; ℓ_ρ denotes the transform

$$\ell_\rho(r) = \int_0^r (\rho')^{-1}(s)\,\mathrm{d}s$$

where $(\rho')^{-1}(r)$ stands for the inverse function of $\mathrm{d}\rho(r)/\mathrm{d}r$. Suppose that there exists a matrix-valued function $R_2(x) = R_2^T(x) > 0$ such that the control law

$$u = \alpha(x) = -R_2^{-1}(x)\big(L_{g_2}V\big)^T \tag{7}$$

globally asymptotically stabilizes (6) with respect to $V(x)$. Then the control law

$$u = \alpha^*(x) = \beta\alpha(x) = -\beta R_2^{-1}(x)\big(L_{g_2}V\big)^T \tag{8}$$

with $\beta \geq 2$ solves the inverse optimal gain assignment problem for the nonlinear system (1) by minimizing the cost functional

$$J_a(u) = \sup_{d\in\mathcal{D}}\left(\lim_{t\to\infty}\left\{2\beta V(x(t)) + \int_0^t \left[l(x) + u^T R_2(x)u - \beta\lambda\rho\left(\frac{|d|}{\lambda}\right)\right]\mathrm{d}\tau\right\}\right)$$

for any $\lambda \in (0, 2]$, where $\mathcal{D}$ is the set of locally bounded functions of x, and

$$l(x) = -2\beta\Big[L_f V + \ell_\rho\big(2\big|L_{g_1}V\big|\big) - L_{g_2}VR_2^{-1}\big(L_{g_2}V\big)^T\Big] + \beta(2-\lambda)\ell_\rho\big(2\big|L_{g_1}V\big|\big) + \beta(\beta-2)L_{g_2}VR_2^{-1}\big(L_{g_2}V\big)^T$$

For nonlinear control systems with disturbances or in the case of trajectory tracking, integral-input-to-state stability (iISS)[19] is a useful theoretical tool to analyze the stability of the closed-loop system. We say that the control system (1) with control inputs $u = k(x)$ is 0-GAS if the 0-disturbance system $\dot{x} = f(x) + g_2(x)k(x)$ is globally asymptotically stable (GAS). We say that the control system (1) with control inputs $u = k(x)$ is iISS with respect to d if for some functions α, $\gamma \in \mathcal{K}_\infty$ and $\beta \in \mathcal{KL}$, for all initial states $x(0)$ and all d, the following estimate holds:

$$\alpha(|x(t)|) \leq \beta(|x(0)|, t) + \int_0^t \gamma(|d(s)|)\,\mathrm{d}s, \qquad \forall t \geq 0$$

The control system (1) with the control law $u = k(x)$ is iISS iff it is 0-GAS and zero-output dissipative, that is, if there exist a positive definite radially unbounded smooth function $V : \mathcal{R}^n \to \mathcal{R}$ and a class $\mathcal{K}_\infty$ function ν such that

$$\dot{V}(t, x) \leq \nu(|d|), \qquad \forall x, d$$

III. Attitude-Tracking Problem Formulation

The spacecraft is assumed to be a rigid body with actuators that provide torques about three mutually perpendicular axes that define a body-fixed frame $\mathcal{B}$. The equations of motion of the spacecraft are given by (Ref. 20, Chap. 4)

$$\dot{q}_v = \tfrac{1}{2}[q_4 I_3 + S(q_v)]w, \qquad \dot{q}_4 = -\tfrac{1}{2}q_v^T w$$
$$\dot{w} = -J^{-1}S(w)Jw + J^{-1}u + J^{-1}d \tag{9}$$

where $q_v \in \mathcal{R}^3$ and $q_4 \in \mathcal{R}$ satisfy $q_v^T q_v + q_4^2 = 1$, $q = [q_v^T, q_4]^T$ denotes the unit quaternion that represents the orientation of the spacecraft in $\mathcal{B}$ with respect to an inertial frame $\mathcal{I}$, w denotes the inertial angular velocity of the spacecraft with respect to the inertial frame $\mathcal{I}$ and expressed in the body frame, $J = J^T$ denotes the positive definite inertia matrix of the spacecraft, $u \in \mathcal{R}^3$ and $d \in \mathcal{R}^3$ denote the control torque and the external disturbance torque, respectively, and I_3 is a 3×3 identity matrix. The notation $S(a)$ induced by a vector $a = [a_1, a_2, a_3]^T$ denotes a skew-symmetric matrix

$$S(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

that satisfies the following important properties:

$$S^T(a) = -S(a), \qquad S(a)b = -S(b)a, \qquad S(a)a = 0$$
$$\|S(a)\| = |a|, \qquad S(a)S(b) = ba^T - a^T b I_3$$
$$S(S(a)b) = ba^T - ab^T \tag{10}$$

These properties are very helpful in the derivation of nonlinear error equations and in designing a nonlinear H_∞ optimal-control law.

Our objective in attitude control is to track an attitude target trajectory to achieve an attitude maneuver with satisfactory accuracy. Let the desired attitude motion of the spacecraft be described by the target unit quaternion $q_c = [q_{cv}^T, q_{c4}]^T$ and the desired angular velocity w_c. From Eqs. (9) we obtain the following differential equations:

$$\dot{q}_{cv} = \tfrac{1}{2}[q_{c4}I_3 + S(q_{cv})]w_c, \qquad \dot{q}_{c4} = -\tfrac{1}{2}q_{cv}^T w_c \tag{11}$$

According to quaternion multiplication (Ref. 20, Appendix A), the quaternion error vector $q_e = [q_{ev}^T, q_{e4}]^T$ that lies in $\mathcal{S}^3$ space and satisfies the algebraic constraint $q_{ev}^T q_{ev} + q_{e4}^2 = 1$ can be expressed by

$$q_{ev} = q_{c4}q_v - S(q_{cv})q_v - q_4 q_{cv}, \qquad q_{e4} = q_{cv}^T q_v + q_4 q_{c4} \tag{12}$$

The rate error w_e is defined as

$$w_e = w - w_c \tag{13}$$

Applying Eqs. (10–13) to Eqs. (9), we can obtain the differential error equations for the tracking-problem formulation:

$$\dot{q}_{ev} = \tfrac{1}{2}[q_{e4}I_3 + S(q_{ev})]w_e + S(q_{ev})w_c, \qquad \dot{q}_{e4} = -\tfrac{1}{2}q_{ev}^T w_e$$
$$J\dot{w}_e = -[S(w_e + w_c)Jw_e + S(w_e)Jw_c] + u + [d - J\dot{w}_c - S(w_c)Jw_c] \tag{14}$$

For simplicity of notation, we let $\epsilon = q_{ev}$ and $\eta = q_{e4}$ and define the extended disturbance as

$$\hat{d}(w_e, t) = \begin{bmatrix} Jw_c \\ d_c - S(w_c)Jw_e - S(w_e)Jw_c \end{bmatrix} \tag{15}$$

where $d_c = d - J\dot{w}_c - S(w_c)Jw_c$ is a combination of the reference signals $\dot{w}_c(t)$ and $w_c(t)$ and the external disturbance $d(t)$. Hence we can rewrite the nonlinear error differential equations (14) as

$$\begin{bmatrix} \dot{\epsilon} \\ \dot{\eta} \\ \dot{w}_e \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[\eta I_3 + S(\epsilon)]w_e \\ -\frac{1}{2}\epsilon^T w_e \\ -J^{-1}S(w_e)Jw_e \end{bmatrix} + \begin{bmatrix} 0_{33} \\ 0_{13} \\ J^{-1} \end{bmatrix} u + \begin{bmatrix} S(\epsilon)J^{-1} & 0_{33} \\ 0_{13} & 0_{13} \\ 0_{33} & J^{-1} \end{bmatrix} \hat{d} \tag{16}$$

where 0_{33} and 0_{13} are the zero matrices of the indicated dimensions. The attitude-tracking problem is then transformed into the problem of the stabilization of the error system (16) with respect to the extended disturbance $\hat{d}(w_e, t)$. Note that both (ϵ, η) and $(-\epsilon, -\eta)$ represent the same physical attitude orientation.

Lemma 2 (Ref. 21): The two coordinate systems, corresponding to q and q_c, respectively, coincide if and only if the error vector ϵ in Eq. (12) is zero, that is, $\epsilon = 0$.

Assumption 1: The target angular velocity $w_c(t)$ and its derivative $\dot{w}_c(t)$ are bounded for all times $t \geq 0$ with known bounds: that is, there exist known, finite, and positive constants $\bar{c}_{w_1}$ and $\bar{c}_{w_2}$ such that $\sup_{t\geq 0} |w_c(t)| \leq \bar{c}_{w_1}$ and $\sup_{t\geq 0} |\dot{w}_c(t)| \leq \bar{c}_{w_2}$.

Applying Lemma 2, we can state the attitude-tracking control problem as follows.

Definition 1: Under Assumption 1, the attitude-tracking control problem is to find a continuous feedback control $u = u(\epsilon, \eta, w_e)$ such that $\epsilon \to 0$ and $w_e \to 0$ as $t \to \infty$.

Remark 1: For setpoint regulation, $q_c \equiv$ constant, $w_c \equiv 0$, and $\dot{w}_c \equiv 0$, we can use the original system (9) and the error signals (12) and (13) to design the control law.

IV. H_∞ Optimal Attitude-Tracking Control

Now we proceed to design the attitude trajectory tracking control law. Define the state

$$x = \begin{bmatrix} \epsilon^T & \eta & w_e^T \end{bmatrix}^T \tag{17}$$

Because η is not an independent variable for the attitude-control system, we also write

$$\tilde{x} = \begin{bmatrix} \epsilon^T & w_e^T \end{bmatrix}^T \tag{18}$$

We choose the Lyapunov function V of the form

$$V(x) = \tfrac{1}{2} w_e^T J w_e + b w_e^T J \epsilon + 2c(1-\eta) \tag{19}$$

where b should be small enough to ensure that V is positive definite and $c > 0$.

Note that $2(1-\eta) = |\epsilon|^2 + (1-\eta)^2$. A sufficient condition for V being positive definite is that

$$Q_V = \begin{bmatrix} 2cI_3 & bJ \\ bJ & J \end{bmatrix} > 0, \qquad \text{that is,} \qquad 2cI_3 > b^2 J \tag{20}$$

Therefore, along the trajectory of Eq. (16) and applying the properties (10) of the skew-symmetric matrix $S(a)$, we have the following Lie derivatives of V:

$$L_f V = (b/2) w_e^T J[\eta I_3 + S(\epsilon)] w_e + c\epsilon^T w_e - b\epsilon^T S(w_e) J w_e$$

$$= (b/2) w_e^T J[\eta I_3 - S(\epsilon)] w_e + c\epsilon^T w_e$$

$$L_{g_1} V = \begin{bmatrix} b w_e^T J S(\epsilon) J^{-1}, \; w_e^T + b\epsilon^T \end{bmatrix}$$

$$L_{g_2} V = w_e^T + b\epsilon^T$$

Before presenting an inverse optimal control law in Theorem 3, we first propose a PD controller that stabilizes an auxiliary system (23) of the nonlinear attitude system (16) on $(\mathcal{S}^3 \times \mathcal{R}^3) \setminus (0, -1, 0)$ by the following theorem.

Theorem 2: The PD control law

$$u = -R_2^{-1}(x)(L_{g_2} V)^T = -\left[k_1 + \left(k_2 / \gamma^2\right)\right](w_e + b\epsilon) \tag{21}$$

with the matrix $R_2(x)$ being

$$R_2^{-1}(x) = \left[k_1 + \left(k_2/\gamma^2\right)\right] I_3 \tag{22}$$

globally asymptotically stabilizes the auxiliary system

$$\begin{bmatrix} \dot{\epsilon} \\ \dot{\eta} \\ \dot{w}_e \end{bmatrix} = \begin{bmatrix} \frac{1}{2}[\eta I_3 + S(\epsilon)] w_e \\ -\frac{1}{2}\epsilon^T w_e \\ -J^{-1} S(w_e) J w_e \end{bmatrix} + \begin{bmatrix} 0_{33} \\ 0_{13} \\ J^{-1} \end{bmatrix} u + \frac{1}{\gamma^2} \begin{bmatrix} S(\epsilon) J^{-1} & 0_{33} \\ 0_{13} & 0_{13} \\ 0_{33} & J^{-1} \end{bmatrix} (L_{g_1} V)^T \tag{23}$$

about the equilibrium point $(\epsilon, \eta, w_e) = (0, 1, 0)$ on $(\mathcal{S}^3 \times \mathcal{R}^3) \setminus (0, -1, 0)$ if the controller gains in Eqs. (19) and (21) satisfy the following conditions:

$$b > 0, \qquad c = 2b\left[k_1 + (k_2 - 1)/\gamma^2\right]$$
$$k_1 > (b/2)\lambda_j + (b^2/\gamma^2)\lambda_j^2 \lambda_i^2 - (k_2 - 1)/\gamma^2$$
$$1 \leq k_2 \leq 1 + b^2 \lambda_j^2 \lambda_i^2 \tag{24}$$

Furthermore, if the initial condition satisfies

$$\eta(0) \geq -1 + (1/2c)\left[\tfrac{1}{2} w_e^T(0) J w_e(0) + b w_e^T(0) J \epsilon(0)\right] + (1/2c)\delta_\eta \tag{25}$$

where δ_η is a sufficiently small constant, $\delta_\eta > 0$, then ϵ and $w_e \to 0$ exponentially.

Proof: If we consider a class $\mathcal{K}_\infty$ function $\rho(r) = \gamma^2 r^2$, it follows that $\rho'(r) = 2\gamma^2 r$, $(\rho')^{-1}(r) = r/2\gamma^2$,

$$\ell_\rho(r) = \int_0^r (\rho')^{-1}(s)\, \mathrm{d}s = \frac{r^2}{4\gamma^2}$$

and $\ell_\rho(2r) = r^2/\gamma^2$. We can then construct an auxiliary system as follows:

$$\dot{x}(t) = f(x) + (1/\gamma^2) g_1(x) \left(L_{g_1} V\right)^T + g_2(x) u \tag{26}$$

which is the state representation of Eq. (23).

Since $\epsilon^T\epsilon + \eta^2 = 1$, it can be shown that $\|\eta I_3 - S(\epsilon)\| = 1$ and

$$\left| w^T J[\eta I_3 - S(\epsilon)] w \right| \leq \|J\| |w|^2$$

Applying the previous inequality and the properties (10) of the skew-symmetric matrix $S(a)$ along the solution of Eq. (23), we have

$$\begin{aligned}
\dot{V} &= L_f V + \frac{1}{\gamma^2}(L_{g_1} V)(L_{g_1} V)^T + L_{g_2} V u \\
&= \left(\frac{b}{2}\right) w_e^T J[\eta I_3 - S(\epsilon)] w_e + c\epsilon^T w_e \\
&\quad + \frac{1}{\gamma^2} \left| \begin{bmatrix} -bJ^{-1} S(\epsilon) J w_e \\ w_e + b\epsilon \end{bmatrix} \right|^2 + (w_e + b\epsilon)^T u \\
&= \frac{b}{2} w_e^T J[\eta I_3 - S(\epsilon)] w_e + c\epsilon^T w_e + \frac{1}{\gamma^2} |w_e + b\epsilon|^2 \\
&\quad - \frac{b^2}{\gamma^2} w_e^T J S(\epsilon) J^{-2} S(\epsilon) J w_e + (w_e + b\epsilon)^T u \\
&\leq \frac{b}{2} w_e^T J[\eta I_3 - S(\epsilon)] w_e + \left(c + \frac{2b}{\gamma^2}\right) \epsilon^T w_e \\
&\quad + \frac{1}{\gamma^2}\left(|w_e|^2 + b^2|\epsilon|^2\right) + \frac{b^2}{\gamma^2} |w_e|^2 \|J\|^2 \|S(\epsilon)\|^2 \|J^{-1}\|^2 \\
&\quad + (w_e + b\epsilon)^T u \leq \frac{b}{2}\lambda_j |w_e|^2 + \left(c + \frac{2b}{\gamma^2}\right)\epsilon^T w_e \\
&\quad + \frac{1}{\gamma^2}\left(|w_e|^2 + b^2|\epsilon|^2\right) + \frac{b^2}{\gamma^2}\lambda_j^2 \lambda_i^2 |\epsilon|^2 |w_e|^2 + (w_e + b\epsilon)^T u \\
&\leq \left(\frac{b}{2}\lambda_j + \frac{1}{\gamma^2} + \frac{b^2}{\gamma^2}\lambda_j^2\lambda_i^2\right)|w_e|^2 + \frac{b^2}{\gamma^2}|\epsilon|^2 \\
&\quad + \left(c + \frac{2b}{\gamma^2}\right)\epsilon^T w_e + (w_e + b\epsilon)^T u
\end{aligned} \tag{27}$$

We design the control law $u(x)$ to be of a PD form, given by Eq. (21), and select the controller parameters b, c, k_1, k_2 satisfying the constraint (24). Clearly, such parameters guarantee Eq. (20) because

$$2cI_3 \geq 4bk_1 I_3 > 2b^2\|J\|I_3 > b^2 J$$

Hence, the Lyapunov function V in Eq. (19) is positive definite and

$$\dot{V} \leq \left[(b/2)\lambda_j + 1/\gamma^2 + (b^2/\gamma^2)\lambda_j^2\lambda_i^2\right]|w_e|^2 + (b^2/\gamma^2)|\epsilon|^2$$
$$+ (c + 2b/\gamma^2)\epsilon^T w_e - \left(k_1 + k_2\big/\gamma^2\right)|w_e + b\epsilon|^2 \leq -\lambda_b|\tilde{x}|^2$$

where $\lambda_b > 0$ is defined by

$$\lambda_b = \min\Big\{k_1 - (b/2)\lambda_j - (b^2/\gamma^2)\lambda_j^2\lambda_i^2 + (k_2 - 1)\big/\gamma^2, b^2(k_2 - 1)\big/\gamma^2 + k_1 b^2\Big\}$$

It follows from Barbalat's theorem (Ref. 22, p. 192) that $\tilde{x}(t) \to 0$ as $t \to \infty$. By Lemma 2, this corresponds to zero orientation error and zero rate error.

Because $|\epsilon|^2 + \eta^2 = 1$, the derivative value $\dot{V} = 0$ implies two equilibrium points $(\epsilon, \eta, w_e) = (0, \pm 1, 0)$ on $\mathcal{S}^3 \times \mathcal{R}^3$, standing for the same attitude orientation. However, it is clear that $(0, -1, 0)$ is an unstable equilibrium point, because it is a local maximum of $V(\epsilon, \eta, w_e)$ on $\mathcal{S}^3 \times \mathcal{R}^3$ and $\dot{V} < 0$ whenever $|\tilde{x}| \neq 0$. We therefore conclude that $\eta \to 1$ as $t \to \infty$ whenever the initial condition $x(0) \neq (0, -1, 0)$ and the PD control law (21) results in the global asymptotic stability of the auxiliary system (23) on $(\mathcal{S}^3 \times \mathcal{R}^3)\backslash(0, -1, 0)$.

If the initial condition (25) is satisfied, we have

$$V(x(t)) \leq V(x(0)) \leq 4c - \delta_\eta = V((0, -1, 0)) - \delta_\eta$$

implying that $x(t)$ is bounded away from the unstable equilibrium point $(0, -1, 0)$ for all $t \geq 0$. Then there exist finite coefficients $c_\eta > 0$, $\lambda_1 > 0$, and $\lambda_2 > 0$ such that

$$(1 - \eta)^2 \leq c_\eta \epsilon^T\epsilon, \qquad V(x) \geq \lambda_1|\tilde{x}|^2$$

$$V(x) \leq \tfrac{1}{2}w_e^T J w_e + b w_e^T J\epsilon + (c + c_\eta)|\epsilon|^2 \leq \lambda_2|\tilde{x}|^2$$

Therefore, it follows that

$$\dot{V} \leq -(\lambda_b/\lambda_2)V$$

Hence, by the comparison principle (Ref. 22, Lemma 2.5) and from $V(x) \geq \lambda_1|\tilde{x}|^2$, we can conclude that $\epsilon \to 0$ and $w_e \to 0$ exponentially. □

Remark 2: Given any initial conditions, the control law (21) will make the errors ϵ and w_e of the auxiliary system (23) converge to zero asymptotically, but $x(t)$ might be arbitrarily close to $(\epsilon, \eta, w_e) = (0, -1, 0)$ for some t before it converges to the desired equilibrium point $(0, 1, 0)$. On the other hand, $(0, -1, 0)$ corresponds to an unstable equilibrium point because any small perturbation will cause a rotation of 360 deg to $\eta = +1$. This situation is avoided if condition (25) is satisfied by choosing k_1 large enough (because $c = 2bk_1$ if $k_2 = 1$) or $w_e(0) = 0$. In fact, in most attitude-tracking control applications, the initial angular velocity $w_e(0)$ is zero or sufficiently small [by choosing the initial angular velocity $w_c(0)$ to be the actual initial velocity $w(0)$], which brings the system (23) to converge to $(\epsilon, \eta, w_e) = (0, 1, 0)$ exponentially.

Note that the state penalty function $l(x)$ in Theorem 1 can be positive semidefinite, which also corresponds to a meaningful cost functional without loss of inverse optimality. Applying Theorems 1 and 2, we will obtain the following theorem on designing the inverse optimal attitude-tracking control law.

Theorem 3: If we let $\beta = \lambda = 2$, then the state-feedback PD control law

$$u = \beta\alpha(x) = -2\left[k_1 + \left(k_2\big/\gamma^2\right)\right](w_e + b\epsilon) \tag{28}$$

with the parameters b, c, k_1, k_2, and $R_2(x)$ given in Theorem 2 solves the inverse optimal gain assignment problem for the attitude-tracking control problem with respect to the extended disturbance $\hat{d}(w_e, t)$ by minimizing the cost functional

$$J_a(u) = \sup_{\hat{d}\in\mathcal{D}}\left\{\lim_{t\to\infty}\left[4V(x(t)) + \int_0^t \left(l(x) + u^T R_2(x)u - \gamma^2|\hat{d}|^2\right)\mathrm{d}\tau\right]\right\} \tag{29}$$

where $l(x)$ is a positive semidefinite state-penalty function defined by

$$l(x) = -4L_f V - (4/\gamma^2)\left|L_{g_1}V\right|^2 + 4L_{g_2}VR_2^{-1}\left(L_{g_2}V\right)^T \tag{30}$$

Furthermore, the control law (28) is also H_∞-optimal for the closed-loop attitude system with respect to the extended disturbance $\hat{d}(w_e, t)$ and the H_∞ performance index (29).

Proof: This theorem is a consequence of Theorems 1 and 2. From the derivations in Theorem 2, we observe that $R_2(x)$ is positive definite and

$$l(x) \geq 4\left[k_1 - (b/2)\lambda_j - (b^2/\gamma^2)\lambda_j^2\lambda_i^2 + (k_2 - 1)/\gamma^2\right]|w_e|^2 + 4b^2\left[k_1 + (k_2 - 1)\big/\gamma^2\right]|\epsilon|^2 \tag{31}$$

which shows that $l(x)$ is positive semidefinite on $\mathcal{S}^3 \times \mathcal{R}^3$ (precisely, $l(x)$ is positive definite in ϵ and w_e). Therefore, $J_a(u)$ is a meaningful cost functional for the attitude-tracking control problem, penalizing both the tracking errors ϵ and w_e, as well as the control effort u and the extended disturbance $\hat{d}$. Substituting $l(x)$ into the cost functional $J_a(u)$, we will get the optimal cost $J_a(u) = 4V(x(0))$ and the "worse-case" extended disturbance (see Ref. 16 for the detailed computations)

$$\hat{d}^*(x) = \lambda(\rho')^{-1}\left(2\left|L_{g_1}V\right|\right)\left[\left(L_{g_1}V\right)^T\big/\left|L_{g_1}V\right|\right] = (2/\gamma^2)\left[L_{g_1}V(x)\right]^T \tag{32}$$

The class $\mathcal{K}_\infty$ function ρ whose derivative ρ' is also a class $\mathcal{K}_\infty$ function is defined in the same way as in the proof of Theorem 2; that is, $\rho(r) = \gamma^2 r^2$, $(\rho')^{-1}(r) = r/2\gamma^2$.

From $J_a(u) = 4V(x(0))$, it follows that, for all $T \geq 0$,

$$\int_0^T \left[l(x) + u^T R_2(x)u\right]\mathrm{d}t \leq \gamma^2\int_0^T |\hat{d}|^2\,\mathrm{d}t + 4V(x(0)) \tag{33}$$

which implies that the closed-loop system for the feedback-control law (28) has $\mathcal{L}_2$ gain less than or equal to γ from the extended disturbance $\hat{d}(w_e, t)$ to the block vector of tracking errors $\tilde{x}(t)$ and control inputs u; thus the disturbance attenuation is achieved. In addition, the Lyapunov function candidate $V(x)$ solves the following Hamilton–Jacobi–Isaacs partial differential equation:

$$\frac{\partial V(x)}{\partial x}f(x) + \frac{\partial V(x)}{\partial x}\left[\frac{1}{\gamma^2}g_1(x)g_1^T(x) - g_2(x)R_2^{-1}g_2^T(x)\right]\frac{\partial V^T(x)}{\partial x} + \frac{1}{4}l(x) = 0$$

Therefore we conclude that the inverse optimal PD control law (28) shows H_∞ optimality with respect to the extended disturbance $\hat{d}(w_e, t)$ and minimizes the H_∞ performance index (29).

In addition, if $\hat{d}(w_e, t) \in \mathcal{L}_2[0, \infty)$, it follows that $\epsilon \in \mathcal{L}_2[0, \infty)$, $w_e \in \mathcal{L}_2[0, \infty)$, and w_c and $\dot{w}_c$ are bounded. By Barbalat's lemma (Ref. 22, p. 192) we conclude that $\epsilon \to 0$ and $w_e \to 0$ as $t \to \infty$; thus the attitude-tracking control problem is solved with global convergence. □

It can be seen from inequality (31) that the state penalty function $l(x)$ in the performance index (29) can be written as $l(x) = \tilde{x}^T Q(x)\tilde{x}$, where the state weighting matrix $Q(x)$ is positive definite.

The system error certainly depends on the gains k_1 and γ. As shown in the performance analysis in Sec. V, the error is approximately proportional to the magnitude of k_1 and a smaller γ brings a smaller system error. Therefore, if we are to reduce the system error, we can increase the magnitude of k_1 and decrease that of γ in the weight $Q(x)$, producing a bigger control effort. Conversely, if k_1 is reduced, then a smaller control effort and a bigger error will result.

The H_∞ inverse optimal control (28) is not model-sensitive because it does not employ feedforward compensation to cancel the quadratic nonlinearities regarding w_c in the model, namely $S(w_e)Jw_c$, $S(w_c)Jw_e$, and $S(w_c)Jw_c$, which are considered as part of the extended disturbance $\hat{d}(w_e, t)$ given by Eq. (15) and will be analyzed in Sec. V. Therefore, the control law (28) does not require complete information on the inertia matrix J, ensuring that the controller is fairly robust to parametric uncertainties. Note that the controller gains of Eq. (28) require only the largest eigenvalue of the inertia matrix J, which is always available or can be easily estimated in practice even when the complete inertia matrix is unknown.

For nonlinear systems with disturbances or in the case of trajectory tracking, iISS[19] is a useful theoretical tool to analyze the stability of the closed-loop system; see Ref. 19 for examples. The stability of the attitude-tracking control system (16) under the PD control law (28) is summarized as follows.

Lemma 3: The attitude trajectory tracking full-state (ϵ, η, w_e)-system under the inverse optimal PD control law of Eq. (28) is integral-input-to-state stable (iISS) on $(\mathcal{S}^3 \times \mathcal{R}^3) \setminus (0, -1, 0)$ with respect to the extended disturbance $\hat{d}(w_e, t)$.

Proof: Employing the feedback-control law (28) in the error equation (16), we obtain the derivative value $\dot{V}(x, t)$ by

$$\begin{aligned}\dot{V} &= L_f V(x) + L_{g_1} V \hat{d} - 2 L_{g_2} V R_2^{-1}(x) \big(L_{g_2} V\big)^T \\ &= -\tfrac{1}{4} l(x) - (1/\gamma^2) \big|L_{g_1} V(x)\big|^2 + L_{g_1} V(x) \hat{d}(t) \\ &\quad - L_{g_2} V(x) R_2^{-1}(x) \big[L_{g_2} V(x)\big]^T\end{aligned}$$

Using Young's inequality,[23] it follows that

$$L_{g_1} V \hat{d} \le (\gamma^2/4)|\hat{d}|^2 + (1/\gamma^2)\big|L_{g_1} V\big|^2$$

where the equals sign is satisfied only when $\hat{d}(w_e, t) = \hat{d}^*(x) = (2/\gamma^2)[L_{g_1} V(x)]^T$. Therefore,

$$\dot{V} \le -\tfrac{1}{4} l(x) - L_{g_2} V R_2^{-1}(x) \big(L_{g_2} V\big)^T + (\gamma^2/4)|\hat{d}(t)|^2 \tag{34}$$

Note that $l(x)$ and $R_2(x)$ are both positive definite on $(\mathcal{S}^3 \times \mathcal{R}^3)\setminus(0, -1, 0)$. As we analyzed in the proof of Theorem 2, the full-state system is 0-GAS on $(\mathcal{S}^3 \times \mathcal{R}^3)\setminus(0, -1, 0)$ if the extended disturbance $\hat{d} = 0$. It is zero-output dissipative because, if we let the output be $h(x) = 0$, then $\dot{V} < (\gamma^2/4)|\hat{d}|^2$. Hence, it is iISS with respect to the extended disturbance $\hat{d}(w_e, t)$. □

V. Performance Analysis

Although the inverse optimal PD control law (28) guarantees the property of iISS of the closed-loop attitude-tracking control system, it does not provide global asymptotic stability due to the presence of extended disturbance. In this section, we will introduce the concept of performance limitation from Ref. 18 to analyze the performance of the attitude trajectory tracking controller and establish tuning guidelines for the selection of the controller gains.

The extended disturbance $\hat{d}(w_e, t)$ defined by Eq. (15) can be represented as

$$\hat{d}(w_e, t) = H(t) w_e + h(t) \tag{35}$$

where

$$H(t) = \begin{bmatrix} 0 \\ -S(w_c)J + S(Jw_c) \end{bmatrix}, \qquad h(t) = \begin{bmatrix} Jw_c \\ d_c \end{bmatrix}$$

Besides Assumption 1, the following assumption is needed to obtain an upper bound of the Euclidean norm $|\hat{d}(w_e, t)|$.

Assumption 2: The external disturbance $d(t)$ is bounded for all time $t \ge 0$ with a known bound; that is, there exists a known, finite, and positive constant $\bar{c}_d$ such that $\sup_{t \ge 0} |d(t)| \le \bar{c}_d$.

The requirement for the external disturbance $d(t)$ in Assumption 2 is rather minimal. We emphasize that the H_∞ inverse optimal control (28) is not restricted to disturbances with

$$\int_0^\infty |d(t)|^2 \,\mathrm{d}t < \infty, \qquad \int_0^\infty |\hat{d}(w_e, t)|^2 \,\mathrm{d}t < \infty$$

because any bounded (and persistent) extended disturbances, consisting of the external disturbance d and the tracking reference signals w_c and $\dot{w}_c$, are allowed in the H_∞ inverse optimality approach[16] and the iISS analysis.[19] The control law (28) guarantees the boundedness of the tracking errors ϵ and w_e for any bounded extended disturbance $\hat{d}(w_e, t)$.

Under Assumptions 1 and 2, there exist finite positive time-varying coefficients c_1, c_2, c_3, β_1, and β_2 such that

$$\begin{aligned}|\hat{d}(w_e, t)|^2 &= w_e^T (H^T H) w_e + 2 w_e^T (H^T h) + (h^T h) \\ &\le c_1 |w_e|^2 + c_2 |w_e| + c_3 \le \beta_1 |w_e|^2 + \beta_2\end{aligned} \tag{36}$$

Such coefficients can be chosen as

$$c_1 = \|H\|^2 \le 4\lambda_j^2 |w_c|^2, \qquad c_2 = 2|H^T h| \le 4\lambda_j |w_c||d_c|$$

$$c_3 = |h^T h| = |Jw_c|^2 + |d_c|^2, \qquad \beta_1 = 6\lambda_j^2 |w_c|^2$$

$$\beta_2 = 3|d_c|^2 + |Jw_c|^2$$

As analyzed in Ref. 18, the inequality (36) implies that $\rho_o(|\hat{d}(w_e, t)|) \le |w_e|$, where $\rho_o(\cdot)$ is not a class $\mathcal{K}_\infty$ function in the case of trajectory tracking or in the presence of external disturbance because it is increasing but not strictly increasing when $c_3 \ne 0$ or $\beta_2 \ne 0$.

If there exists no external disturbance, $d(t) = 0$, then the 0-GAS property holds for the setpoint regulation problem (corresponding to $\dot{w}_c = w_c = 0$ and $q_c = \text{constant}$) because $c_1 = c_2 = c_3 = 0$ and then $\dot{V} < 0$. However, the static PD controller (28) cannot guarantee the GAS either in the trajectory tracking or in the existence of external disturbance. This fact brings about a performance limitation of the inverse optimal PD controller. The control performance is determined by the gain values of the controller. Therefore, it is important to set up a relation between the gain values and the system errors, which is found by examining points that satisfy $\dot{V} = 0$.

Theorem 4: Choose $k_2 = 1$. Suppose that λ_c is the minimum eigenvalue of the matrix

$$Q_c = \begin{bmatrix} \left(2k_1 + \dfrac{1}{\gamma^2}\right) b^2 I_3 & \left(k_1 + \dfrac{1}{\gamma^2}\right) b I_3 \\ \left(k_1 + \dfrac{1}{\gamma^2}\right) b I_3 & \left(2k_1 + \dfrac{1}{\gamma^2} - \dfrac{b}{2}\lambda_j - \dfrac{b^2}{\gamma^2}\lambda_j^2 \lambda_i^2 - \dfrac{\gamma^2}{4}\bar{c}_1\right) I_3 \end{bmatrix}$$

Let the performance limitation $|\tilde{x}|_{\mathrm{PL}}$ be defined as the Euclidean norm of $\tilde{x}$ that satisfies $\dot{V} = 0$. If the inverse optimal PD control law (28) with $k_2 = 1$ is applied to the attitude trajectory tracking system (16) and $\lambda_c > 0$ is satisfied, then its performance limitation is upper bounded by

$$|\tilde{x}|_{\mathrm{PL}} \le \frac{\gamma^2}{8\lambda_c} \left[\bar{c}_2 + \sqrt{\bar{c}_2^2 + \frac{16}{\gamma^2}\lambda_c \bar{c}_3}\right] \tag{37}$$

where $\bar{c}_1 = \sup\{c_1(t)\}$, $\bar{c}_2 = \sup\{c_2(t)\}$ and $\bar{c}_3 = \sup\{c_3(t)\}$, with c_1, c_2, and c_3 defined by Eq. (36).

Proof: Substituting $l(x)$ of Eq. (30) with $k_2 = 1$ into $\dot{V}$ in inequality (34), we have

$$\begin{aligned}\dot{V} &\leq -\left[k_1 - (b/2)\lambda_j - (b^2/\gamma^2)\lambda_j^2\lambda_i^2\right]|w_e|^2 - k_1 b^2|\epsilon|^2 \\ &\quad - \left(k_1 + 1/\gamma^2\right)|w_e + b\epsilon|^2 + (\gamma^2/4)|\hat{d}|^2 \\ &\leq -x^T Q_c x + (\gamma^2/4)(c_2|w_e| + c_3) \\ &\leq -\lambda_c|x|^2 + (\gamma^2/4)\bar{c}_2|x| + (\gamma^2/4)\bar{c}_3 \end{aligned} \tag{38}$$

where λ_c is the minimum eigenvalue of Q_c. By the definition of the performance limitation, inequality (38) brings about the performance limitation of inequality (37). □

Remark 3: Either in the case of trajectory tracking or in the presence of external disturbances, c_1, c_2, and c_3 are not always zero and thus there exists a bound for the system error. Inequality (37) can be considered as an upper bound of the system error for all time and thus can be used as a formula to predict the performance of the closed-loop system for various values of the controller gains.

Remark 4: Because the right-hand side of inequality (37) is monotonically decreasing with λ_c, the inequality holds if λ_c is replaced by any smaller positive value. The minimum eigenvalue λ_c of the matrix Q_c in Theorem 4 satisfies

$$\lambda_c \geq \min\left\{b^2 k_1, \left[k_1 - (b/2)\lambda_j - (b^2/\gamma^2)\lambda_j^2\lambda_i^2 - (\gamma^2/4)\bar{c}_1\right]\right\} \tag{39}$$

This can be seen by writing Q_c as a sum of two terms,

$$Q_c = \begin{bmatrix} k_1 b^2 I_3 & 0 \\ 0 & \left(k_1 - \dfrac{b}{2}\lambda_j - \dfrac{b^2}{\gamma^2}\lambda_j^2\lambda_i^2 - \dfrac{\gamma^2}{4}\bar{c}_1\right)I_3 \end{bmatrix} + \left(k_1 + \frac{1}{\gamma^2}\right)\begin{bmatrix} I_3 & 0 \\ \dfrac{1}{b}I_3 & I_3 \end{bmatrix}\begin{bmatrix} b^2 I_3 & bI_3 \\ 0 & 0 \end{bmatrix}$$

and noting that the minimum eigenvalue of the second term is zero.

Remark 5: Suppose that $|w_c(t)| \leq \bar{c}_4$, where $\bar{c}_4 = \sup\{|w_c(t)|\}$ is a positive constant. Then it follows from Eq. (36) that $\beta_1 \leq \bar{\beta}_1 = 6\bar{c}_4^2\lambda_j$ and

$$|\hat{d}|^2 \leq \beta_1|w_e|^2 + \beta_2 \leq \bar{\beta}_1|w_e|^2 + \beta_2$$

Substituting these into $\dot{V}$ in relation (34), we have

$$\begin{aligned}\dot{V} &\leq -\left[k_1 - (b/2)\lambda_j - (b^2/\gamma^2)\lambda_j^2\lambda_i^2\right]|w_e|^2 - k_1 b^2|\epsilon|^2 \\ &\quad - \left(k_1 + 1/\gamma^2\right)|w_e + b\epsilon|^2 + (\gamma^2/4)|\hat{d}|^2 \\ &\leq -\left[k_1 - (b/2)\lambda_j - (b^2/\gamma^2)\lambda_j^2\lambda_i^2 - (\gamma^2/4)\bar{\beta}_1\right]|w_e|^2 \\ &\quad - k_1 b^2|\epsilon|^2 + (\gamma^2/4)\beta_2 \end{aligned} \tag{40}$$

As shown in Eq. (36), β_2 consists of $w_c(t)$, $\dot{w}_c(t)$, and $d(t)$. If

$$\left[k_1 - (b/2)\lambda_j - (b^2/\gamma^2)\lambda_j^2\lambda_i^2 - (\gamma^2/4)\bar{\beta}_1\right] > 0$$

it can be deduced from relation (40) that the closed-loop system is β_2-to-(ϵ, w_e) stable. In addition, if $w_c(t)$, $\dot{w}_c(t)$, and $d(t) \in \mathcal{L}_2[0, \infty)$, then the $\mathcal{L}_2$ gain from β_2 to (ϵ, w_e) is finite and ϵ, $w_e \in \mathcal{L}_2[0, \infty)$.

After analyzing the performance estimates of the closed-loop system with the inverse optimal controller (28), we present some guidelines for gain selection. Without loss of generality, we choose $k_2 = 1$. The parameter b is chosen small enough to assure that the Lyapunov function V of Eq. (19) is positive definite. By appropriately choosing k_1 and b, we can choose $\lambda_c \geq b^2 k_1$ as follows.

Suppose that $|w_c| \leq \bar{c}_4$. It follows that $c_1 \leq \bar{c}_1 = 4\lambda_j^2\bar{c}_4^2$. If we choose the gain k_1 to satisfy the condition

$$k_1 > (\gamma^2/4)\bar{c}_1 = \gamma^2\lambda_j^2\bar{c}_4^2 \geq (\gamma^2/4)c_1 \tag{41}$$

and the inequality

$$k_1 - (b/2)\lambda_j(-b^2/\gamma^2)\lambda_j^2\lambda_i^2 - (\gamma^2/4)\bar{c}_1 \geq b^2 k_1\lambda_j^2\lambda_i^2$$

then it follows that

$$\lambda_c \geq b^2 k_1 \tag{42}$$

(because $\lambda_j\lambda_i \geq 1$) and the following constraint of the parameter b:

$$b \leq (1/2A)\left\{-\lambda_j/2 + \sqrt{\lambda_j^2\big/4 + 4A\left[k_1 - (\gamma^2/4)\bar{c}_1\right]}\right\} \tag{43}$$

where $A = (k_1 + 1/\gamma^2)\lambda_j^2\lambda_i^2$.

From Eq. (41), we note that (k_1/γ^2) should be larger as the maximum of $|w_c|^2$ goes, which means that $(\sqrt{k_1}/\gamma)$ should be directly proportional to $\bar{c}_4$; that is,

$$\sqrt{k_1}/\gamma \propto \bar{c}_4 \tag{44}$$

If we further choose k_1 large enough or γ small enough so that $4A(k_1 - \gamma^2/4)\bar{c}_1 \gg \lambda_j^2/4$, the constraint (43) can be approximated by

$$b \leq \frac{1}{\sqrt{A}}\sqrt{k_1 - \frac{\gamma^2}{4}\bar{c}_1} = \sqrt{\frac{k_1 - \gamma^2\lambda_j^2\bar{c}_4^2}{\left(k_1 + 1/\gamma^2\right)\lambda_j^2\lambda_i^2}}$$

Together with relation (44), we could choose

$$b \propto \sqrt{k_1\Big/\left[\left(k_1 + 1/\gamma^2\right)\lambda_j^2\lambda_i^2\right]} \tag{45}$$

with a small scale that is less than 1, which implies that the gain b is slowly increasing with γ and k_1 but not in a perfectly proportional way. When γ is small enough so that $1/\gamma^2 \gg k_1$, it follows that

$$b \propto \gamma\sqrt{k_1}\big/\lambda_j\lambda_i$$

Using the inequality

$$\sqrt{x_1^2 + x_2^2} \leq |x_1| + |x_2| \qquad \text{for all} \qquad x_1, x_2 \in R$$

and substituting inequality (42) into inequality (37), we can represent the performance limitation (37) as follows:

$$|\tilde{x}|_{\mathrm{PL}} \leq \left(\gamma^2\big/8b^2k_1\right)\left[2\bar{c}_2 + (4/\gamma)\sqrt{b^2k_1\bar{c}_3}\right]$$

which implies that

$$|\tilde{x}|_{\mathrm{PL}} \leq (\bar{c}_2/4)(1/b)^2\left(\gamma\big/\sqrt{k_1}\right)^2 + \left(\sqrt{\bar{c}_3}\big/2\right)(1/b)\left(\gamma\big/\sqrt{k_1}\right) \tag{46}$$

from which we can conclude that the performance limitation $|\tilde{x}|_{\mathrm{PL}}$ in inequality (37) can be considered as a performance estimate with respect to the magnitudes of the gains k_1 and γ: For a fixed γ, a bigger k_1 results in a smaller error $\tilde{x}$. A smaller γ also brings a smaller error $\tilde{x}$. A small $\mathcal{L}_2$ gain γ affects the performance by increasing the attenuation of the external disturbance and reference inputs.

Therefore, conditions (41) and (44–46) establish the relations between the attitude-tracking error $\tilde{x}$ and the controller parameters k_1, γ, and b. Hence, we can consider conditions (44) and (45) as selection guidelines for the gains k_1 and b in the optimal control law (28).

VI. Simulation Results

In this section, a rigid-body microsatellite is considered to demonstrate the performance of the H_∞ inverse optimal tracking controller.

In the simulations, we assume that the inertia matrix of the satellite is

$$J = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 8 \end{bmatrix} \text{kg}\cdot\text{m}^2$$

with $\lambda_j = 10$ and $\lambda_i = \lambda_{\max}(J^{-1}) = 0.125$. We also assume that the external disturbance $d(t)$ is given by

$$d(t) = \begin{bmatrix} 0.005 - 0.05\sin(2\pi t/400) + \delta(200, 0.2) + v_1 \\ 0.005 + 0.05\sin(2\pi t/400) + \delta(250, 0.2) + v_2 \\ 0.005 - 0.03\sin(2\pi t/400) + \delta(300, 0.2) + v_3 \end{bmatrix} \text{Nm} \tag{47}$$

where $\delta(T, \Delta T)$ denotes an impulsive disturbance with magnitude 1, period T, and width ΔT. The terms v_1, v_2, and v_3 denote white Gaussian noises with mean values $m_{v_1} = m_{v_2} = m_{v_3} = 0$ and variances $\sigma_{v_1}^2 = \sigma_{v_2}^2 = \sigma_{v_3}^2 = 0.005^2$. The desired angular velocity w_c of the spacecraft is given by

$$w_c = \begin{bmatrix} 0.05\sin(2\pi t/400) \\ -0.05\sin(2\pi t/400) \\ 0.03\sin(2\pi t/400) \end{bmatrix} \text{rad/s} \tag{48}$$

with $\bar{c}_4 = \sup\{|w_c(t)|\} = 0.077$ and is plotted in Fig. 1 as dotted lines. The target quaternion q_c can then be computed by integrating Eq. (11) with the initial condition $q_c(0) = [0,\ 0,\ 0,\ 1]^T$.

Further, the initial conditions of the quaternion q and the angular velocity w are given by $q(0) = [0.3, 0.2, 0.3, -0.8832]^T$ and $w(0) = [0, 0, 0]^T$. The gain k_2 in the controller (28) is assumed to be $k_2 = 1$. The gain k_1 is chosen so that $k_1 \gg \gamma^2\lambda_j^2\bar{c}_4^2$; the value of b is then obtained from relations (43) and (45).

First, we choose a set of gains, $\gamma = 1$, $k_1 = 4.0$, $k_2 = 1$, and $b = 0.18$, to demonstrate the tracking performance of the derived H_∞ optimal PD tracking controller (28). Figures 1 and 2 depict the time responses of the angular velocities w and w_c and the corresponding rate error w_e, from which we can say that the actual angular velocity w tracks the target angular velocity w_c well with a small error. The sharp peaks in Figs. 1 and 2 are due to large impulsive disturbances. Figure 3 plots the time behavior of the actual quaternion

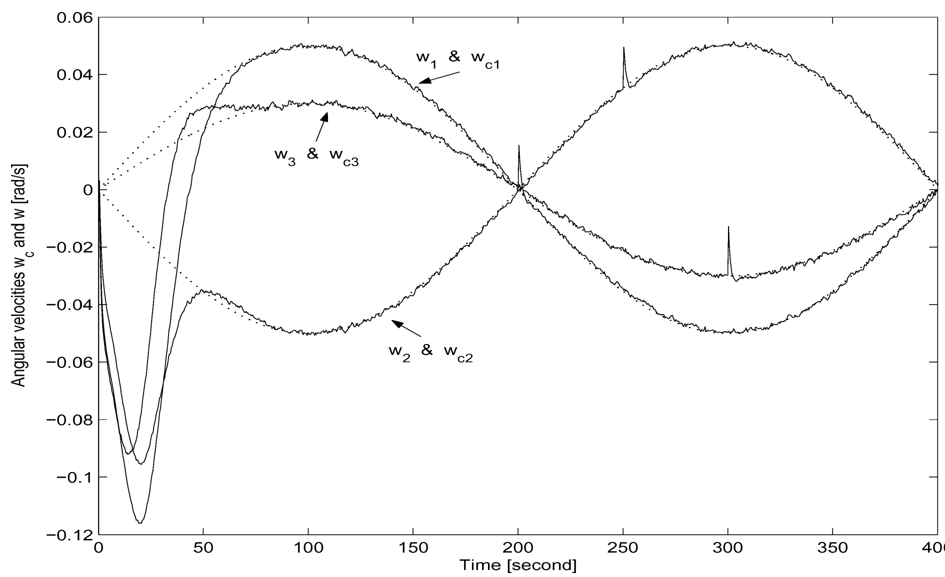

Fig. 1 Time responses of the angular velocities *w* and w_c: . . . , target angular velocity w_c and ——, *w*.

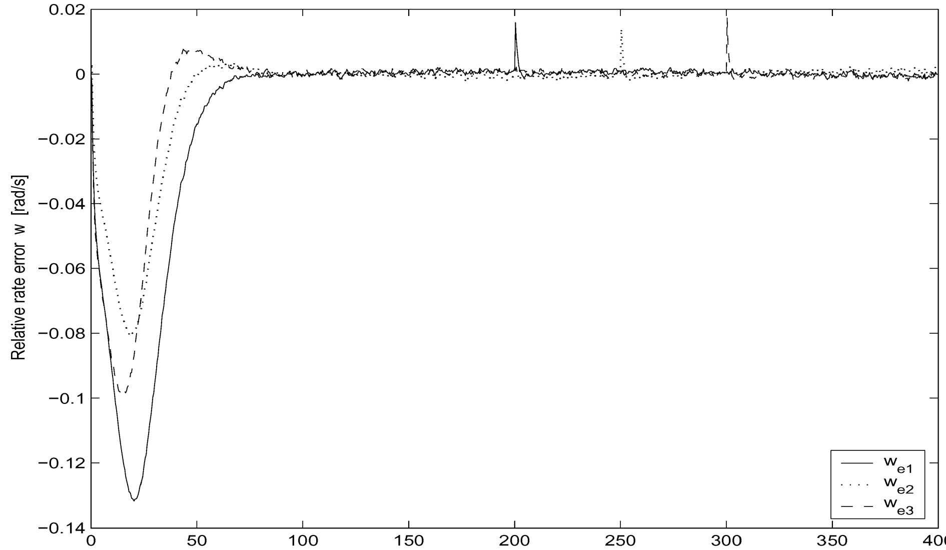

Fig. 2 Time history of the rate error $w_e = [w_{e1}, w_{e2}, w_{e3}]^T$.

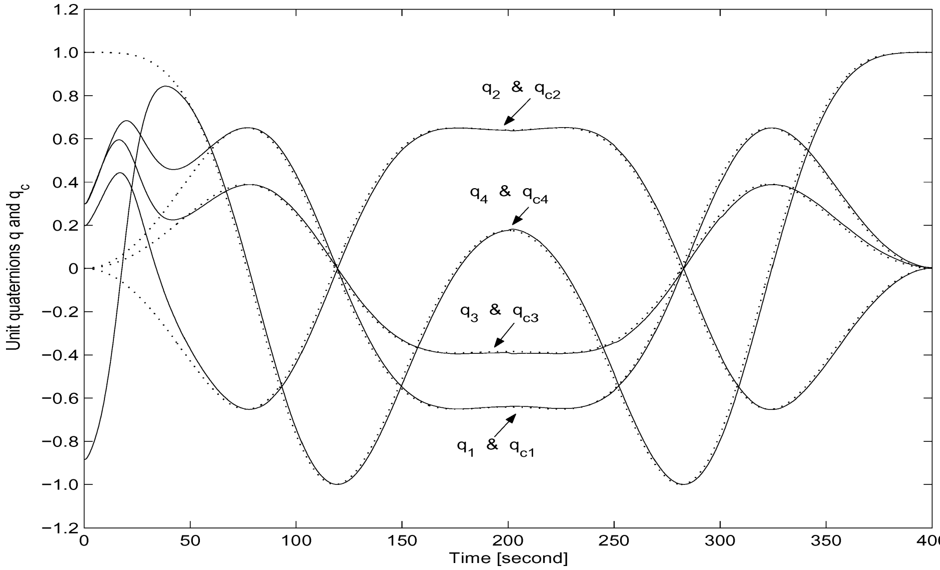

Fig. 3 Time responses of the unit quaternions q and q_c: . . . , target quaternion q_c and ——, q.

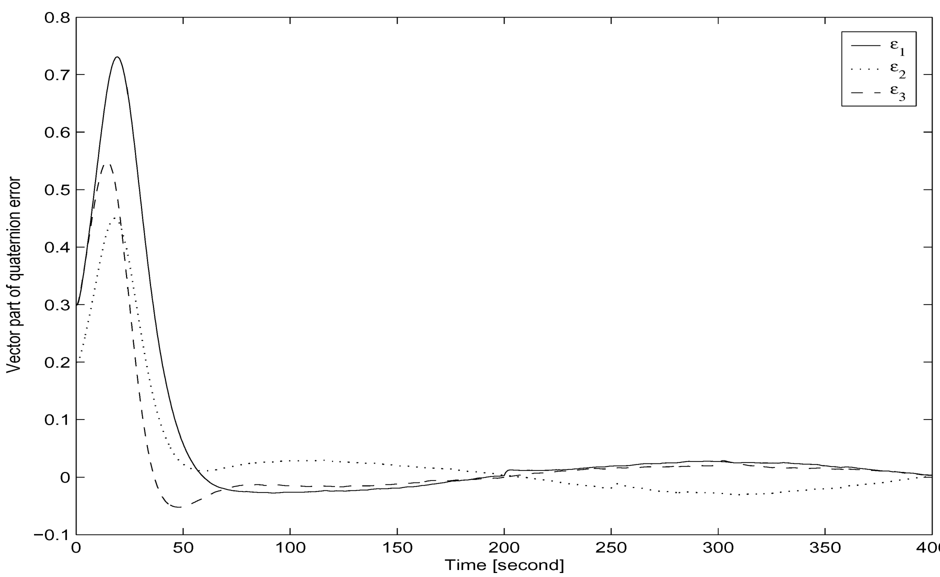

Fig. 4 Time histories of the tracking error $\epsilon = [\epsilon_1, \epsilon_2, \epsilon_3]^T$.

q compared to the desired quaternion q_c and Fig. 4 shows the orientation error ϵ. It is seen from Figs. 3 and 4 that the PD tracking law (28) achieves good performance in attitude tracking with satisfactory orientation error. Figure 5 shows the time history of the control effort $u = [u_1, u_2, u_3]^T$.

Next, we fix the $\mathcal{L}_2$ gain $\gamma = 1.0$ and change the magnitude of the gain k_1 from 2.0 to 4.0 and finally to 8.0, that is, increased by a factor of 2 each time, to show the relationship between the gain k_1 and the system error. After conditions (43) and (45) are applied, the value of b is changed accordingly as 0.164, 0.180, and 0.190. Under control law (28) and with these values of k_1 and b, the closed-loop system enters into steady tracking within 100 s as before. The norm of the tracking error $|\tilde{x}| = \sqrt{(\epsilon^T \epsilon + w_e^T w_e)}$ and the state ϵ_1 in the interval [100, 800] s for different values of the gain k_1 are depicted as in Figs. 6 and 7, in which the norm $|\tilde{x}|$ and the steady error ϵ_1 approximately decrease by half each time. A larger gain k_1 results in a smaller tracking error $\tilde{x}$. For example, as k_1 increases from $2.0 \rightarrow 4.0 \rightarrow 8.0$, the maximum value of the norm $|\tilde{x}|$ is reduced from $0.088 \rightarrow 0.048 \rightarrow 0.025$ and the maximum of ϵ_1 from $0.050 \rightarrow 0.028 \rightarrow 0.015$. The history of the control effort u_1 in the interval [0, 80] s is plotted as in Fig. 8. (A shorter interval is selected to see the effects of the gain k_1 on the control torque u clearly.) From Fig. 8 we note that a larger k_1 results in a bigger control effort to achieve the steady attitude tracking in shorter time.

Last, we fix the gain $k_1 = 4$ and change the magnitude of the $\mathcal{L}_2$ gain γ from 2.0 to 1.0 and finally to 0.5, that is, decreased by a factor of 2 each time, to show the relationship between the $\mathcal{L}_2$ gain γ and the system error $\tilde{x}$. The value of b is changed accordingly as 0.194, 0.180, and 0.142 by the tuning rule (45). With the controller (28), the closed-loop system achieves steady tracking within 100 s as before. The norm of the error $|\tilde{x}|$ and the attitude error ϵ_1 in the interval [100, 800] s are plotted as in Figs. 9 and 10, from which it is seen that the steady tracking error $\tilde{x}$ is reduced as γ decreases. For example, as γ is reduced from 2.0 to 1.0 and then to 0.5, the maximum of the norm $|\tilde{x}|$ is reduced from 0.050 to 0.046 and then to 0.036. The time history of the control input u_1 in the interval [0, 80] s is plotted as in Fig. 11, from which we observe that a smaller γ results in a bigger control effort to achieve steady attitude tracking in a shorter time.

Compared with the existing techniques for attitude control, one advantage of our approach is the flexible tradeoff between the tracking performance and the control effort, which usually leads to more

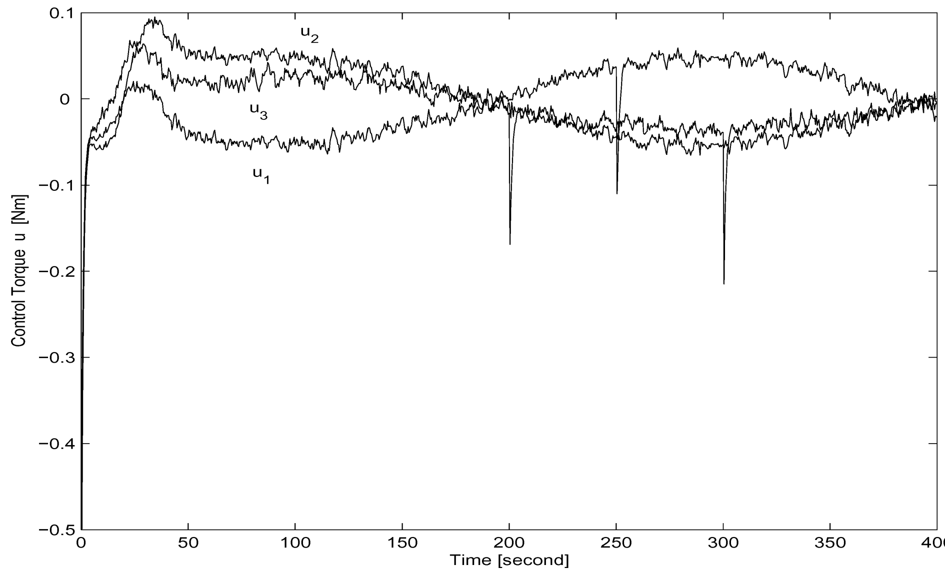

Fig. 5 Time history of the control effort $u = [u_1, u_2, u_3]^T$.

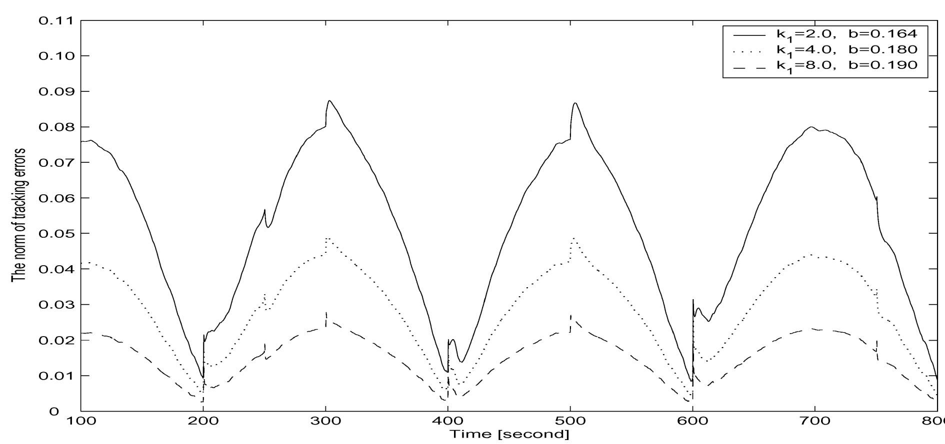

Fig. 6 Time history of the norm $|\tilde{x}|$ in the interval [100, 800] s, $\gamma = 1$.

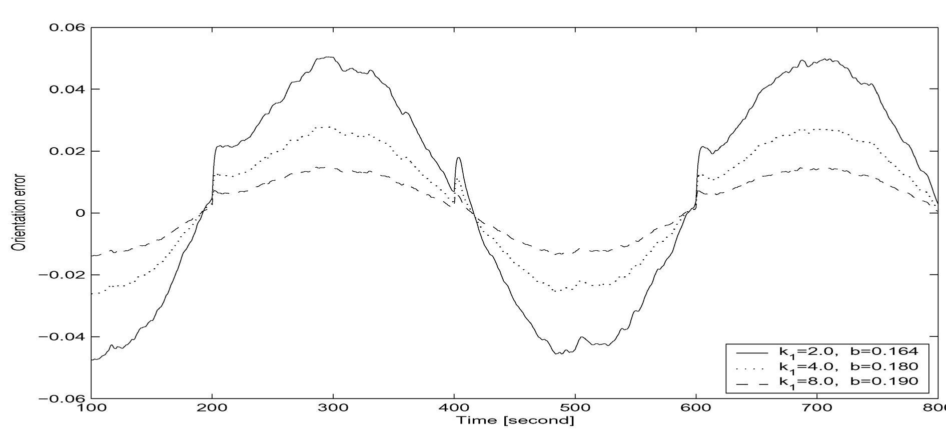

Fig. 7 Tracking error ϵ_1 in the interval [100, 800] s, $\gamma = 1$.

economical control for a good tracking performance. To further illustrate this point, a computer simulation is presented to compare our H_∞ inverse optimal controller with the variable structure controller (VSC) in Ref. 24, where the attitude-tracking problem was also considered with external disturbances, and the VSC was proved to be globally stable under certain conditions. The VSC in the current simulation is given by

$$u = -u_m \operatorname{sgn}(s), \qquad s = w_e + k\epsilon$$

$$\dot{k} = -\gamma_k u_m \sum_{i=1}^{3} [\operatorname{sgn}(k)|\epsilon_i| + \epsilon_i \operatorname{sgn}(s_i)] \tag{49}$$

where $u_m = 0.07$ Nms, $\gamma_k = 0.001$, and $k(0) = 0.105$, respectively. For our H_∞ inverse optimal control (28), we let $b = 0.10$, $k_1 = 10$, and $\gamma = 0.045$. The initial conditions are $w(0) = [0, 0, 0]^T$ and $q(0) = [0, 0, 0, 1]^T$. Figure 12 plots the norm of the tracking error $|\tilde{x}|$ and the control effort u, from which we see that the tracking

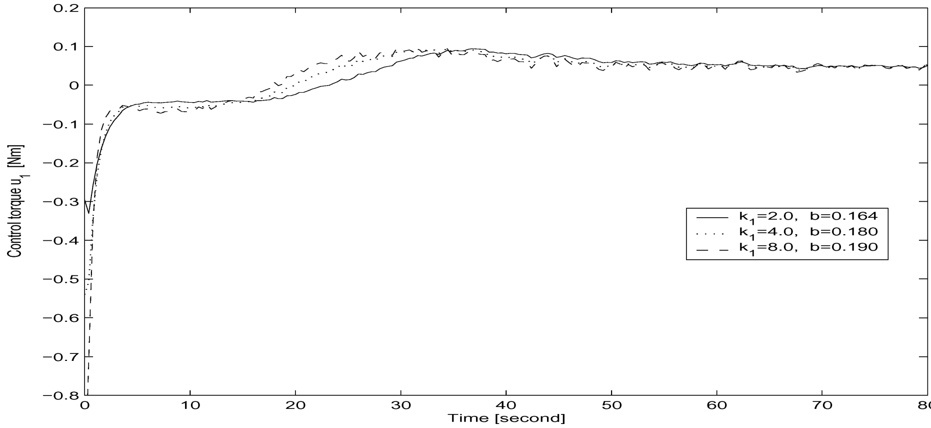

Fig. 8 Control effort u_1 in the interval [0, 80] s, $\gamma = 1$.

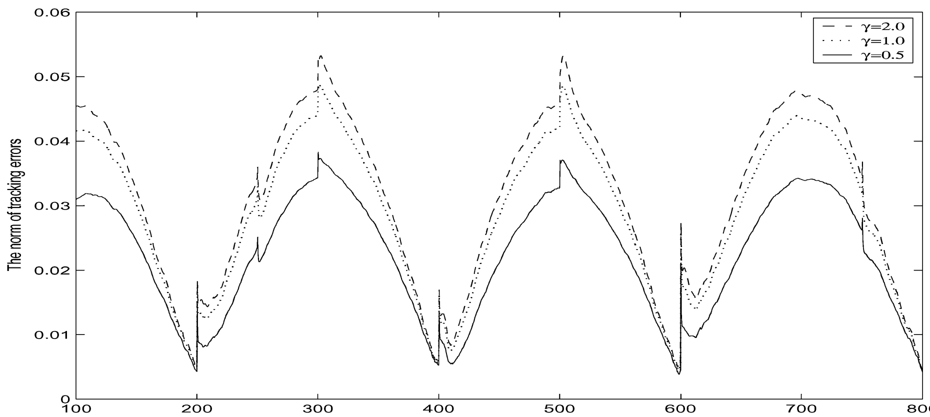

Fig. 9 Time history of the norm $|\tilde{x}|$ in the interval [100, 800] s, $k_1 = 4$.

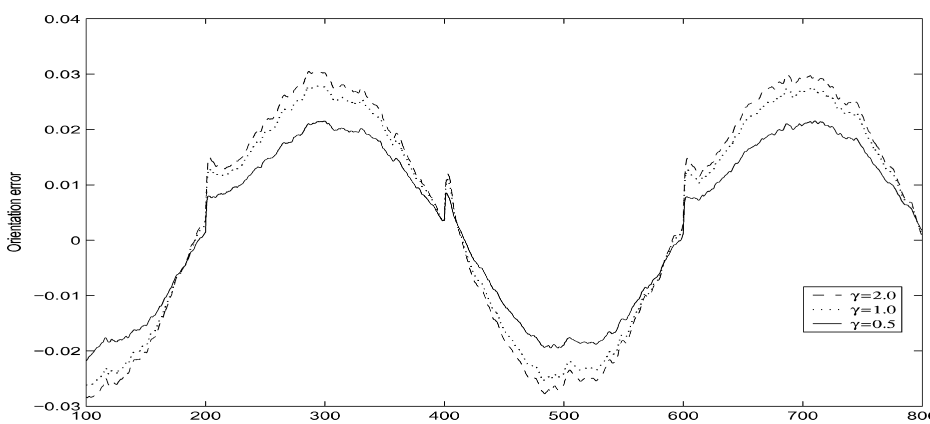

Fig. 10 Tracking error ϵ_1 in the interval [100, 800] s, $k_1 = 4$.

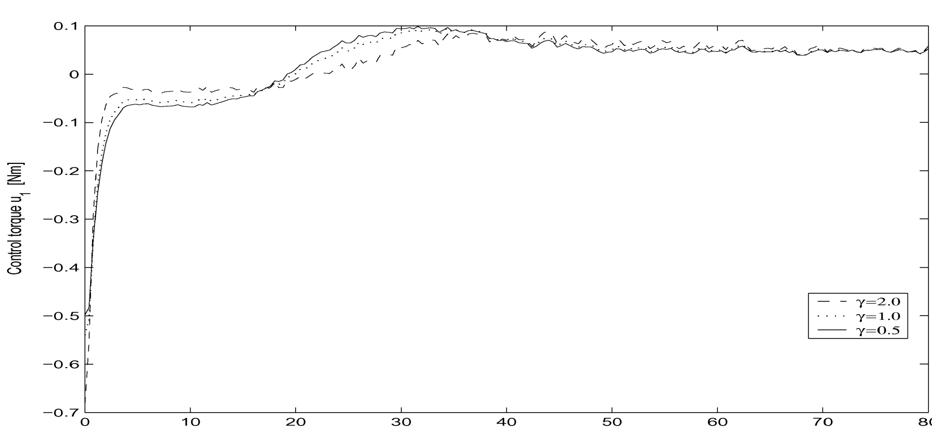

Fig. 11 Control effort u_1 in the interval [0, 80] s, $k_1 = 4$.

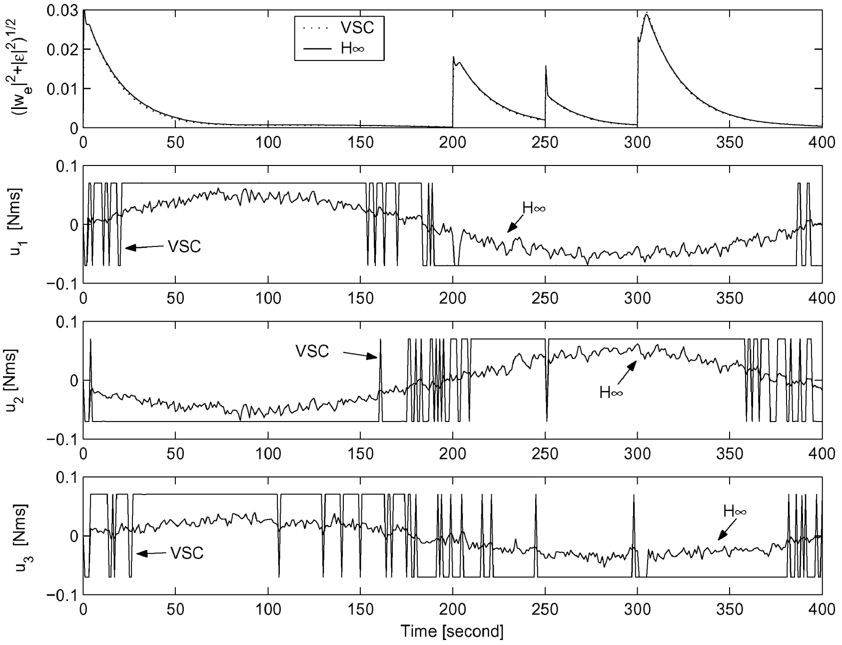

Fig. 12 Comparison between VSC and H_∞ inverse optimal control.

performances of the two controllers are nearly identical. The VSC (49), by switching the control u between $\pm u_m$, is producing a torque that has the same effect as the control torque of the H_∞ inverse optimal control (28). Of course, in this way the VSC would require a control power of $|u_m|^2$, which is much higher than that required by the H_∞ inverse optimal controller. Furthermore, simulations showed that u_m cannot be smaller than 0.07 if the VSC (49) is to be able to track the reference (48) subject to the disturbance (47).

Note that for a fair comparison, the control torques of the H_∞ inverse optimal controller and the VSC were both under the same bound $u_m = 0.07$ in the simulations, as shown in Fig. 12.

VII. Conclusions

With the introduction of extended disturbances, the robust inverse optimal control method has been applied to the attitude-tracking control problem of a rigid spacecraft with external disturbances. The proposed state-feedback control law is robust inverse optimal with respect to a meaningful cost functional that includes penalties on tracking errors and control efforts as well as extended disturbances. The associated control Lyapunov function solves a Hamilton–Jacobi–Isaacs partial differential equation. Thus, nonlinear H_∞ optimality with respect to extended disturbances is achieved without obtaining a direct solution of the HJI equation and the disturbance is also attenuated. Such a state-feedback law is in the form of a PD controller, which is easy to implement in practice. Performance estimates have been given in terms of the performance limitation. Based on the performance analysis, tuning rules have been established as selection guidelines for the proportional and derivative gains. Numerical simulations have been carried out to verify the performance analysis and the validity of the tuning rules.

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